

CONSUMER DEMAND WITH SEVERAL LINEAR CONSTRAINTS: A GLOBAL ANALYSIS

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1. Introduction

Economists sometimes find themselves in the position of having to extend the neoclassical model of consumer demand to settings where, in addition to the conventional budget constraint, there are one or more additional linear constraints that restrict the consumer's utility maximization problem. Examples include point rationing [Tobin-Houthakker (1950-51), Tobin (1952)]; models of time allocation where the time constraint cannot be collapsed into the budget constraint [de Serpa (1971); de Donnea (1972); McConnell (1975); Lyon (1978) Larson and Shaikh (2001)]; and multi-period portfolio allocation problems [Diamond and Yaari (1972)]. Without exception, the existing literature has focused on differential properties of the resulting demand functions-i.e. issues such as the effect of rationing on demand elasticities, the Le Chatelier--Samuelson Theorem, the generalization of the Hick-Slutsky decomposition, and other comparative static results [see Kusumoto (1976), Chichilnisky and Kalman (1978), Hatta (1980), Wan (1981) and the references cited above]. By employing some "tricks with utility functions" in the spirit of Gorman (1976), I am able to obtain a *global* characterization of these demand functions.¹ Specifically, I develop an algorithm for deriving the demand functions that apply when there are M linear constraints from those that apply when there is only a single constraint. The algorithm permits one to derive all of the existing comparative static results in a simple and compact manner. It also has some value for empirical demand analysis, because it shows how to

¹ The literature cited above derives local results for the solution to maximization problems with multiple *non-linear* constraints, while the global results developed here apply only for *linear* constraints. The simpler structure of the constraints makes it possible to obtain the stronger results.

compute the demand functions associated with maximization problems involving multiple linear constraint based on direct or indirect utility functions associated with known conventional demand functions.

The paper is organized as follows. Section 2 presents some preliminary results which are needed for the main analysis, but are also of interest as "tricks" in their own right. Section 3 considers the utility maximization problem with two linear constraints, summarizes the existing comparative static results, develops the new Global Representation Theorem, and shows how this can be used to derive and sharpen the existing comparative static results. Section 4 considers a utility maximization problem with three linear constraints and develops the analogous Global Representation Theorem for the solution to this problem in terms of known demand functions associated with a conventional single-constraint problem; the results developed here provide the basis for extension to problems involving more than three linear constraints. Section 5 offers some concluding observations.

2. Preliminaries

Consider the conventional utility maximization problem

$$\underset{x_1, \dots, x_N}{\text{maximize}} \quad u(x) \quad \text{s.t.} \quad \sum_{i=1}^N p_i x_i = y \quad (1)$$

where $u(x)$ is a conventional neoclassical utility function, i.e. twice continuously differentiable, increasing and strictly quasi-concave. Let $x_i = h^i(p, y)$, $i = 1, \dots, N$, be the ordinary demand functions associated with (1) and let $v(p, y)$ be the corresponding indirect utility function. Next consider a utility maximization problem like (1) but involving a different utility function, denoted $\tilde{u}(x)$, which however is related to $u(x)$ as specified in (2b). This utility maximization is:

$$\underset{x_1, \dots, x_N}{\text{maximize}} \quad \tilde{u}(x) \quad \text{s.t.} \quad \sum_1^N p_i x_i = y \quad (2a)$$

where

$$\tilde{u}(x) = u\left(x_1, \dots, x_{N-1}, x_N - \sum_1^{N-1} \psi_j x_j\right) \quad (2b)$$

for an arbitrary set of constants $\psi_1, \dots, \psi_{N-1}$. Let $x_i = \tilde{h}^i(p, y), i = 1, \dots, N$, be the ordinary demand functions corresponding to (2a, b) and let $\tilde{v}(p, y)$ be the corresponding indirect utility function.

The relationship between the demand functions and indirect utility function associated with (1) and those associated with (2a, b) is given in the following lemma, which is proved in the

Appendix:

Lemma 1. The two sets of demand and indirect utility functions satisfy

$$\tilde{v}(p_1, \dots, p_N, y) = v[p_1 + p_N \psi_1, \dots, p_{N-1} + p_N \psi_{N-1}, p_N, y] \quad (3a)$$

$$\tilde{h}^i(p_1, \dots, p_N, y) = h^i[p_1 + p_N \psi_1, \dots, p_{N-1} + p_N \psi_{N-1}, p_N, y] \quad i = 1, \dots, N-1 \quad (3b)$$

$$\begin{aligned} \tilde{h}^N(p_1, \dots, p_N, y) &= h^N[p_1 + p_N \psi_1, \dots, p_{N-1} + p_N \psi_{N-1}, p_N, y] \\ &+ \sum_1^{N-1} \psi_j h^j[p_1 + p_N \psi_1, \dots, p_{N-1} + p_N \psi_{N-1}, p_N, y] . \end{aligned} \quad (3c)$$

Now consider the utility maximization problem

$$\underset{x_1, \dots, x_N}{\text{maximize}} \quad u(x, z) \quad \text{s.t.} \quad \sum_1^N p_i x_i = y, \quad z \text{ fixed} \quad (4a)$$

where in general z is a vector and $u(\cdot)$ has the conventional properties with respect to (x, z) . Let

$x_j = \hat{h}^j(p, z, y), i = 1, \dots, N$, be the ordinary demand functions associated with (4a) and $\hat{v}(p, z, y)$ the

corresponding indirect utility function. A different but related utility maximization is

$$\underset{x,z}{\text{maximize}} \ u(x,z) \text{ s.t. } \sum_1^N p_i x_i + \sum_1^M q_j z_j = y \quad (4b)$$

where the q_j 's are the prices of the z_j 's. The solution to (4b) involves two sets of ordinary demand functions, $x_i = h_i^x(p, q, y)$, $i = 1, \dots, N$ and $z_j = h_j^z(p, q, y)$, $j = 1, \dots, M$, and an indirect utility function $v(p, q, y)$. The relation between the demand functions and indirect utility functions associated with 4(a) and 4(b) is as follows:

Lemma 2. Given (p, z, y) , define the functions $\hat{q}_j(p, z, y)$, $j = 1, \dots, M$ to be the solution to

$$z_j = h_j^z(p, \hat{q}, y + \sum \hat{q}_j z_j) \quad j = 1, \dots, M. \quad (5)$$

Then,

$$\hat{v}(p, z, y) = v(p, \hat{q}, y + \sum \hat{q}_j z_j) \quad (6a)$$

$$\hat{h}^i(p, z, y) = h_i^x(p, \hat{q}, y + \sum \hat{q}_j z_j) \quad i = 1, \dots, N. \quad (6b)$$

The proof is contained in the Appendix. Given these preliminary results, I now turn to the main analysis.

3. Two Linear Constraints

The maximization in (1) is a conventional problem with a single linear constraint. I now consider what happens when there are two linear constraints:

$$\underset{x}{\text{maximize}} \ u(x) \quad \text{s.t.} \quad \sum_1^N p_i x_i = y \quad \text{and} \quad \sum_1^N w_i x_i = t \quad (7)$$

where $u(x)$ is the same utility function as in (1), and $N > 2$. Let $x_i = \bar{h}^i(p, w, y, t)$, $i = 1, \dots, N$, be the ordinary demand functions associated with (7) and $\bar{v}(p, w, y, t)$ the corresponding indirect utility function. The question to be addressed here is: what is the relationship between the

demand functions and the indirect utility function associated with (7) and those associated with (1)? Specifically, if we know the formulas for $v(p, y)$ and $h^i(p, y)$ that solve (1), can we directly write down the formulas for $\bar{v}(p, w, y, t)$ and $\bar{h}^i(p, w, y, t)$ in (7)?

Before giving the answer, it is useful to set down what *is* known about the functions associated with (7). The following results are derived in Diamond and Yaari (1972) and Lyon (1978), and parallel the conventional properties associated with the solution to (1):

Proposition 1.

- (a) $\bar{v}(\cdot)$ is quasiconvex and homogenous of degree zero in (p, y) .
- (b) $\bar{v}(\cdot)$ is quasiconvex and homogenous of degree zero in (w, t) .
- (c) $\bar{v}(\cdot)$ is decreasing in (p, w) and increasing in (y, t) .
- (d) $\bar{h}^i(\cdot)$ is homogenous of degree zero in (p, y) and also in (w, t) .
- (e) Roy's Identity

$$\frac{\partial \bar{v} / \partial p_i}{\partial \bar{v} / \partial y} = \frac{\partial \bar{v} / \partial w_i}{\partial \bar{v} / \partial t} = -\bar{h}^i(p, w, y, t) \quad i=1, \dots, N. \quad (8)$$

Note that the last result may be rearranged into the form

$$\frac{\partial \bar{v} / \partial p_i}{\partial \bar{v} / \partial w_i} = \frac{\partial \bar{v} / \partial y}{\partial \bar{v} / \partial t} \quad i=1, \dots, N, \quad (9)$$

which can be regarded as a set of restrictions on functional forms eligible to serve as indirect utility functions. The differential properties of the ordinary demand functions are as follows

Proposition 2.

$$(a) \quad \frac{\partial \bar{h}^i}{\partial p_j} = \frac{\partial x_i}{\partial p_j} \Big|_{\text{compensated}} - x_j \frac{\partial \bar{h}^i}{\partial y} \quad i, j = 1, \dots, N.$$

$$(b) \quad \frac{\partial \bar{h}^i}{\partial w_j} = \frac{\partial x_i}{\partial w_j} \Big|_{\text{compensated}} - x_j \frac{\partial \bar{h}^i}{\partial t} \quad i, j = 1, \dots, N.$$

Define the $(N \times N)$ matrices $\bar{S}^p$ and $\bar{S}^w$ with typical elements

$$\bar{S}_{ij}^p \equiv \frac{\partial \bar{h}^i}{\partial p_j} + x_j \frac{\partial \bar{h}^i}{\partial y} \quad \text{and} \quad \bar{S}_{ij}^w \equiv \frac{\partial \bar{h}^i}{\partial w_j} + x_j \frac{\partial \bar{h}^i}{\partial t}. \quad (10)$$

$$(c) \quad \bar{S}^p \text{ and } \bar{S}^w \text{ are each symmetric, so that } \bar{S}_{ij}^p = \bar{S}_{ji}^p \text{ and } \bar{S}_{ij}^w = \bar{S}_{ji}^w.$$

$$(d) \quad \bar{S}^p \text{ and } \bar{S}^w \text{ are each negative semi-definite, so that } \bar{S}_{ii}^p \leq 0 \text{ and } \bar{S}_{ii}^w \leq 0.$$

$$(e) \quad \frac{\partial \bar{v}}{\partial t} \bar{S}_{ij}^p = \frac{\partial \bar{v}}{\partial y} \bar{S}_{ji}^w \quad i, j = 1, \dots, N.$$

Parts (a) and (b) follow from Theorem 2 in Chichilnisky and Kalman (1978). In part (a) the compensation is performed by adjusting y so as to maintain $u(x)$ constant, while in part (b) the compensation is performed by adjusting t . Parts (c), (d), and (e) can be obtained by manipulating Roy's Identity (8) or, alternatively, by specializing the comparative static results in Theorems 6 and 7 of Hatta (1980) and Theorem 1 of Wan (1981).

In order to obtain the global characterization of the functions $\bar{h}^i(\cdot)$ and $\bar{v}(\cdot)$, I first solve the second constraint in (7) for one of the x 's --say, x_N -- as a function of all the others

$$x_N = \frac{t}{w_N} - \sum_{i=1}^{N-1} \frac{w_i}{w_N} x_i \quad (11)$$

and substitute this into the first constraint in (7) to obtain

$$\sum_{i=1}^{N-1} \left(p_i - p_N \frac{w_i}{w_N} \right) x_i = y - p_N \frac{t}{w_N}. \quad (12)$$

Now consider the following utility maximization problem with a *single* constraint

$$\underset{x_1, \dots, x_{N-1}}{\text{maximize}} \quad u \left[x_1, \dots, x_{N-1}, \frac{t}{w_N} - \sum_{i=1}^{N-1} \frac{w_i}{w_N} x_i \right] \quad \text{s.t.} \quad \sum_{i=1}^{N-1} \left(p_i - p_N \frac{w_i}{w_N} \right) x_i = y - p_N \frac{t}{w_N} \quad (P)$$

and denote the resulting ordinary demand functions $x_i = h_i^*(p, w, y, t)$ $i = 1, \dots, N-1$ and the corresponding indirect utility function $u = v^*(p, w, y, t)$. If one defines the function $h_N^*(\cdot)$ by

$$h_N^*(p, w, y, t) = \frac{t}{w_N} - \sum_{i=1}^{N-1} \frac{w_i}{w_N} h_i^*(p, w, y, t) \quad (13)$$

and compares (7) with (P), it is clear that the two problems are equivalent - i.e.,

$$\begin{aligned} \bar{v}(p, w, y, t) &\equiv v^*(p, w, y, t) \\ \bar{h}^i(p, w, y, t) &\equiv h_i^*(p, w, y, t) \quad i = 1, \dots, N. \end{aligned} \quad (14)$$

It is necessary, therefore, to establish a relationship between the solutions to (P) and to (1). For this purpose, I introduce a fourth utility maximization problem

$$\begin{aligned} \underset{x_1, \dots, x_{N-1}, z}{\text{maximize}} \quad & u \left[x_1, \dots, x_{N-1}, z - \sum_{i=1}^{N-1} \frac{w_i}{w_N} x_i \right] \\ \text{s.t.} \quad & \sum_{i=1}^{N-1} \left(p_i - p_N \frac{w_i}{w_N} \right) x_i + qz = y - p_N \frac{t}{w_N} \end{aligned} \quad (D)$$

with solutions $x_i = \tilde{h}^i(p, w, q, y, t)$, $i = 1, \dots, N-1$ and $z = \tilde{h}^N(p, w, q, y, t)$ and associated indirect utility function $\tilde{v}(p, w, q, y, t)$. One can now use Lemma 1 to compare problem (D) with (1) and obtain a relationship between $[\tilde{h}(\cdot), \tilde{v}(\cdot)]$ and $[h(\cdot), v(\cdot)]$. Then, one can use Lemma 2 to compare problem (P) with (D) and obtain a relationship between $[h^*(\cdot), v^*(\cdot)]$ and $[\tilde{h}(\cdot), \tilde{v}(\cdot)]$. In this way, one can compare (P) with (1) and obtain the desired relationship between $[\bar{h}(\cdot), \bar{v}(\cdot)]$ and $[h(\cdot), v(\cdot)]$.

Applying Lemma 1 to (D) and setting $\psi_i = w_i/w_N$ yields the following solutions for the indirect utility and ordinary demand functions:

$$\tilde{v}(p, w, q, y, t) = v \left[\left(p_1 + (q - p_N) \frac{w_1}{w_N} \right), \dots, \left(p_{N-1} + (q - p_N) \frac{w_{N-1}}{w_N} \right), q, y - p_N \frac{t}{w_N} \right] \quad (15a)$$

$$\tilde{h}^i(p, w, q, y, t) = h^i \left[\left(p_1 + (q - p_N) \frac{w_1}{w_N} \right), \dots, \left(p_{N-1} + (q - p_N) \frac{w_{N-1}}{w_N} \right), q, y - p_N \frac{t}{w_N} \right] \quad i = 1, \dots, N-1 \quad (15b)$$

$$\begin{aligned} \tilde{h}^N(p, w, q, y, t) = & h^N \left[\left(p_1 + (q - p_N) \frac{w_1}{w_N} \right), \dots, \left(p_{N-1} + (q - p_N) \frac{w_{N-1}}{w_N} \right), q, y - p_N \frac{t}{w_N} \right] \\ & + \sum_{i=1}^{N-1} \frac{w_i}{w_N} h^i \left[\left(p_1 + (q - p_N) \frac{w_1}{w_N} \right), \dots, \left(p_{N-1} + (q - p_N) \frac{w_{N-1}}{w_N} \right), q, y - p_N \frac{t}{w_N} \right]. \end{aligned} \quad (15c)$$

Next, applying Lemma 2, one observes that, if the quantity $\bar{q}$ satisfies

$$\frac{t}{w_N} = \tilde{h}^N \left(p, w, \bar{q}, y + \bar{q} \frac{t}{w_N}, t \right), \quad (16)$$

then

$$h_i^*(p, w, y, t) = \tilde{h}^i \left(p, w, \bar{q}, y + \bar{q} \frac{t}{w_N}, t \right) \quad i = 1, \dots, N-1 \quad (17a)$$

and

$$v^*(p, w, y, t) = \tilde{v} \left(p, w, \bar{q}, y + \bar{q} \frac{t}{w_N}, t \right). \quad (17b)$$

Substituting (15c) into (16) and multiplying through by w_N yields an equivalent expression

which implicitly defines the function $\bar{q} = \bar{q}(p, w, y, t)$:

$$t = \sum_{i=1}^N w_i h^i \left[\left(p_1 + (\bar{q} - p_N) \frac{w_1}{w_N} \right), \dots, \left(p_{N-1} + (\bar{q} - p_N) \frac{w_{N-1}}{w_N} \right), \bar{q}, y + (\bar{q} - p_N) \frac{t}{w_N} \right]. \quad (18)$$

Combining (13), (14), (15) and (17) completes the proof of the following:

GLOBAL REPRESENTATION THEOREM:

Let $\bar{q}(p, w, y, t)$ satisfy (18). Then,

$$\bar{h}^i(p, w, y, t) = h^i \left[\left(p_1 + (\bar{q} - p_N) \frac{w_1}{w_N} \right), \dots, \left(p_{N-1} + (\bar{q} - p_N) \frac{w_{N-1}}{w_N} \right), \bar{q}, y + (\bar{q} - p_N) \frac{t}{w_N} \right] \quad i = 1, \dots, N. \quad (19a)$$

$$\bar{v}(p, w, y, t) = v \left[\left(p_1 + (\bar{q} - p_N) \frac{w_1}{w_N} \right), \dots, \left(p_{N-1} + (\bar{q} - p_N) \frac{w_{N-1}}{w_N} \right), \bar{q}, y - (\bar{q} - p_N) \frac{t}{w_N} \right]. \quad (19b)$$

This is the desired global representation of the solution to (7).² A convenient simplification of this representation is obtained by introducing

$$\theta(p, w, y, t) \equiv \frac{[\bar{q}(p, w, y, t) - p_N]}{w_N}. \quad (20)$$

Making the corresponding substitution in (18), $\theta(\cdot)$ is defined implicitly by

$$t = \sum_1^N w_i h^i(p_1 + \theta w_1, \dots, p_N + \theta w_N, y + \theta t). \quad (18')$$

Using $\theta(p, w, y, t)$, the representation in (19a,b) becomes

$$x_i = \bar{h}^i(p, w, y, t) = h^i(p_1 + \theta w_1, \dots, p_N + \theta w_N, y + \theta t) \quad i = 1, \dots, N \quad (19'a)$$

$$u = \bar{v}(p, w, y, t) = v(p_1 + \theta w_1, \dots, p_N + \theta w_N, y + \theta t). \quad (19'b)$$

The theorem leads to the following algorithm for constructing the solution to the dual linear constraint problem (7) using known solutions to (1):

² It should be emphasized that the choice of good N in this context is entirely arbitrary, and any of the x_i 's can be used for this purpose provided the corresponding w_i is non-zero. Moreover, the two constraints can be interchanged: the w_i 's and t can be substituted for the p_i 's and y in (18) or (18') and in (19a,b) or (19'a,b).

ALGORITHM

Step 1: Take a system of demand functions $h^i(p, y)$ that is known to solve (1), and solve (18') for

$$\theta = \theta(p, w, y, t).$$

Step 2: Combine $\theta(p, w, y, t)$ with $h^i(p, y)$ and $v(p, y)$ as indicated in (19'a,b) to obtain the demand system and indirect utility function associated with the solution to (7).

Note that an explicit representation of the direct utility function involved in (1) and (7) is *not* required. One can start out with an indirect utility function $v(p, y)$ at:

Step 0: Apply Roy's Identity to $v(p, y)$ to obtain the demand system $h^i(p, y)$.

Several observations about this result should be noted. First, if for some (p', w', y', t') the second constraint in (7) is not separately binding -- i.e., $\sum w'_i h^i(p', y') = t'$ -- then it follows from (18) that

$$\bar{q}(p', w', y', t') = p'_N \quad (21a)$$

and

$$\theta(p', w', y', t') = 0. \quad (21b)$$

Moreover, by differentiating (18), it can be shown that, in the vicinity of this point,

$$\frac{\partial \bar{q}(p', w', y', t')}{\partial t} < 0. \quad (21c)$$

Otherwise, however, $\bar{q} \neq p_N$, $\theta > 0$, and the result in (21c) does not necessarily apply. The interpretation of $\theta(p, w, y, t)$ will become evident below.

Second, the global representation in (19a,b) or (19'a,b) can be used to derive all of the results in Propositions 1 and 2. For example, it is trivial to verify the adding up conditions, $\sum p_i \bar{h}^i(\cdot) = y$ and $\sum w_i \bar{h}^i(\cdot) = t$. Given the homogeneity properties of $v(\cdot)$ and $h^i(\cdot)$, (18) implies

that $\bar{q}(p, w, y, t)$ is homogenous of degree 1 in (p, y) and degree 0 in (w, t) ; it follows from (21) that $\theta(p, w, y, t)$ is homogenous of degree 1 in (p, y) and degree -1 in (w, t) . From (19a,b) or (19'a,b), this ensures the homogeneity of $\bar{v}(\cdot)$ and $\bar{h}^i(\cdot)$. Similarly, the quasiconvexity of $v(\cdot)$ implies that $\bar{v}(\cdot)$ is quasiconvex in (p, y) and (w, t) . Straightforward differentiation of (18) and (19) yields Roy's Identity, from which the negative semi-definiteness and symmetry conditions in parts (c) - (e) of Proposition 2 may be deduced.

It is instructive to derive parts (c) - (e) of Proposition 2 directly from the Global Representation Theorem. Denote the Slutsky terms associated with the demand functions $h^i(\cdot)$ by

$$S_{ij}(p, y) \equiv \frac{\partial h^i(p, y)}{\partial p_j} + h^j(p, y) \frac{\partial h^i(p, y)}{\partial y} \quad (22)$$

whereas, as noted in (10), those associated with the demand functions $\bar{h}^i(\cdot)$ are given by

$$\bar{S}_{ij}^p(p, w, y, t) \equiv \frac{\partial \bar{h}^i(p, w, y, t)}{\partial p_j} + \bar{h}^j(p, w, y, t) \frac{\partial \bar{h}^i(p, w, y, t)}{\partial y}. \quad (23)$$

Differentiation of (19'a) yields the following relationships linking the derivatives of $\bar{h}^i(\cdot)$ with respect to p_j and y to those of $h^i(\cdot)$:

$$\frac{\partial \bar{h}^i(p, w, y, t)}{\partial p_j} = \frac{\partial h^i(\pi, \eta)}{\partial p_j} + \left[\sum w_1 \frac{\partial h^i(\pi, \eta)}{\partial p_1} + t \frac{\partial h^i(\pi, \eta)}{\partial y} \right] \frac{\partial \theta}{\partial p_j} \quad (24a)$$

$$\frac{\partial \bar{h}^i(p, w, y, t)}{\partial y} = \frac{\partial h^i(\pi, \eta)}{\partial y} + \left[\sum w_1 \frac{\partial h^i(\pi, \eta)}{\partial p_1} + t \frac{\partial h^i(\pi, \eta)}{\partial y} \right] \frac{\partial \theta}{\partial y} \quad (24b)$$

where $\pi_i \equiv p_i + \theta w_i$, $i = 1, \dots, N$, and $\eta \equiv y + \theta t$. Hence,

$$\bar{S}_{ij}^p(p, w, y, t) = S_{ij}(\pi, \eta) + \left[\sum w_1 \frac{\partial h^i(\pi, \eta)}{\partial p_1} + t \frac{\partial h^i(\pi, \eta)}{\partial y} \right] \left[\frac{\partial \theta}{\partial p_j} + x_j \frac{\partial \theta}{\partial y} \right]. \quad (25)$$

By using (18'), one obtains:

$$\begin{aligned}
\left[\sum w_i \frac{\partial h^i(\pi, \eta)}{\partial p_i} + t \frac{\partial h^i(\pi, \eta)}{\partial y} \right] &= \left[\sum w_i \frac{\partial h^i(\pi, \eta)}{\partial p_i} + \sum w_i x_i \frac{\partial h^i(\pi, \eta)}{\partial y} \right] \\
&= \sum w_j \left(\frac{\partial h^i(\pi, \eta)}{\partial p_j} + x_j \frac{\partial h^i(\pi, \eta)}{\partial y} \right) \\
&= \sum w_j S_{ij}(\pi, \eta) .
\end{aligned} \tag{26}$$

Implicit differentiation of (18') yields:

$$\begin{aligned}
\left[\frac{\partial \theta}{\partial p_j} + x_j \frac{\partial \theta}{\partial y} \right] &= - \frac{1}{\Delta} \left[\sum w_i \frac{\partial h^i(\pi, \eta)}{\partial p_j} + x_j \sum w_i \frac{\partial h^i(\pi, \eta)}{\partial y} \right] \\
&= - \frac{1}{\Delta} \sum w_i S_{ij}(\pi, \eta)
\end{aligned} \tag{27}$$

where

$$\Delta \equiv \sum \sum w_i w_j S_{ij}(\pi, \eta) \leq 0 . \tag{28}$$

Substituting (26) and (27) into (25) yields the formula for one set of Slutsky terms associated with (7):

$$\bar{S}_{ij}^p(p, w, y, t) = S_{ij}(\pi, \eta) - \frac{[\sum w_j S_{ij}(\pi, \eta)][\sum w_i S_{ij}(\pi, \eta)]}{\Delta} \quad i, j = 1, \dots, N. \tag{29}$$

From (29), it follows that $\bar{S}^p$ is symmetric and negative semi-definite. Similarly, from (19'a)

$$\frac{\partial \bar{h}^i(p, w, y, t)}{\partial w_j} = \theta \frac{\partial h^i(\pi, \eta)}{\partial p_j} + \left[\sum w_i \frac{\partial h^i(\pi, \eta)}{\partial p_i} + t \frac{\partial h^i(\pi, \eta)}{\partial y} \right] \frac{\partial \theta}{\partial w_j} \tag{30a}$$

$$\frac{\partial \bar{h}^i(p, w, y, t)}{\partial t} = \theta \frac{\partial h^i(\pi, \eta)}{\partial y} + \left[\sum w_i \frac{\partial h^i(\pi, \eta)}{\partial p_i} + t \frac{\partial h^i(\pi, \eta)}{\partial y} \right] \frac{\partial \theta}{\partial t}, \tag{30b}$$

and, from implicit differentiation of (18'),

$$\begin{aligned}
\left[\frac{\partial \theta}{\partial w_j} + x_j \frac{\partial \theta}{\partial t} \right] &= - \frac{1}{\Delta} \left[x_j + \sum w_i \frac{\partial h^i(\pi, \eta)}{\partial p_j} \theta - x_j + x_j \sum w_i \frac{\partial h^i(\pi, \eta)}{\partial y} \theta \right] \\
&= - \frac{\theta}{\Delta} \sum w_i S_{ij}(\pi, \eta).
\end{aligned} \tag{31}$$

Hence,

$$\bar{S}_{ij}^w(p, w, y, t) = \theta \bar{S}_{ij}^p(p, w, y, t) \quad i, j = 1, \dots, N. \tag{32}$$

From this it follows that that $\bar{S}^w$ is symmetric and negative semi-definite, which completes the derivation of parts (c) and (d) of Proposition 2.

With regard to part (e) of Proposition 2, differentiation of (19'b) yields

$$\frac{\partial \bar{v}(p, w, y, t)}{\partial t} = \theta \frac{\partial v(\pi, \eta)}{\partial y} + \left[\sum w_j \frac{\partial v(\pi, \eta)}{\partial p_j} + t \frac{\partial v(\pi, \eta)}{\partial y} \right] \frac{\partial \theta}{\partial t} \tag{33a}$$

$$\frac{\partial \bar{v}(p, w, y, t)}{\partial y} = \frac{\partial v(\pi, \eta)}{\partial y} + \left[\sum w_j \frac{\partial v(\pi, \eta)}{\partial p_j} + t \frac{\partial v(\pi, \eta)}{\partial y} \right] \frac{\partial \theta}{\partial y}. \tag{33b}$$

However, using Roy's identity

$$\begin{aligned}
\left[\sum w_j \frac{\partial v(\pi, \eta)}{\partial p_j} + t \frac{\partial v(\pi, \eta)}{\partial y} \right] &= \left[\sum w_j \frac{\partial v(\pi, \eta)}{\partial p_j} + \sum w_j x_j \frac{\partial v(\pi, \eta)}{\partial y} \right] \\
&= \left[\sum w_j \frac{\partial v(\pi, \eta)}{\partial p_j} - \sum w_j \left(\frac{\partial v(\pi, \eta)/\partial p_j}{\partial v(\pi, \eta)/\partial y} \right) \frac{\partial v(\pi, \eta)}{\partial y} \right] \\
&= 0.
\end{aligned}$$

Hence,

$$\theta(p, w, y, t) = \frac{\partial \bar{v}(p, w, y, t)}{\partial t} / \frac{\partial \bar{v}(p, w, y, t)}{\partial y}, \tag{34}$$

which provides an interpretation of θ and a proof that $\theta \geq 0$. Combining (34) with (32) yields part (e) of Proposition 2.

Third, it is important to recognize that, while θ is a function of (p, w, y, t) , it is not an arbitrary function of these variables since it is required to satisfy the homogeneity properties noted above;³ in addition, (27) and (31) imply restrictions on the derivatives of $\theta(p, w, y, t)$.

Fourth, setting $i = j$ in (29) yields a version of the Strong Le Chatelier--Samuelson Principle:

Proposition 3 For every $i = 1, \dots, N$,

$$\bar{S}_{ij}^p(p, w, y, t) = S_{ij}(\pi, \eta) - \frac{[\sum w_l S_{il}(\pi, \eta)]^2}{\Delta} \geq S_{ij}(\pi, \eta).$$

Thus, including the second set of constraints in (7) makes the own price derivative of the compensated demand function for x_i smaller in absolute value than when it is omitted. This result is stronger than Hatta's Theorem 8 for general, non-linear constraints because that yields a Le Chatelier--Samuelson inequality holding only at the vector (p', w', y', t') which satisfies (21a). Proposition 3 shows that, in the case of linear constraints, this inequality holds generally - the own price derivative of the compensated demand function for x_i is smaller in absolute value when the second constraint is added to (7) than when it is omitted.

4. Three Linear Constraints

I now consider what happens when the maximization problem involves three linear constraints:

$$\underset{x}{\text{maximize}} \ u(x) \quad \text{s.t.} \quad \sum p_i x_i = y, \quad \sum w_i x_i = t, \quad \text{and} \quad \sum c_i x_i = k \quad (35)$$

³ These homogeneity properties can also be derived from (34).

where $u(x)$ is the same utility function as in (1) and $N > 3$. Let

$x_i = \bar{h}^i(p, w, c, y, t, k)$, $i = 1, \dots, N$, denote the ordinary demand functions associated with (35), and let $\bar{v}(p, w, c, y, t, k)$ be the corresponding indirect utility function; as before the demand functions and indirect utility function associated with the single-constraint problem, (1), are denoted by $x_i = h^i(p, y)$ and $v(p, y)$.

To simplify (35), one uses the second and third constraints to solve for two of the x 's – say, x_{N-1} and x_N – as functions of the others, yielding:

$$\begin{aligned} x_{N-1} &= \frac{\alpha_{kt}}{\alpha_{N-1}} - \sum_{i=1}^{N-2} \frac{\alpha_i}{\alpha_{N-1}} x_i \\ x_N &= \frac{\beta_{kt}}{\beta_N} - \sum_{i=1}^{N-2} \frac{\beta_i}{\beta_N} x_i \end{aligned} \quad (36)$$

where

$$\begin{aligned} \alpha_i &\equiv w_i c_N - w_N c_i & \alpha_{kt} &\equiv t c_N - w_N k \\ \beta_i &\equiv w_{N-1} c_i - w_i c_{N-1} & \beta_{kt} &\equiv w_{N-1} k - t c_{N-1} . \end{aligned} \quad (37)$$

For future reference, note that

$$\alpha_{N-1} = \beta_N \quad \text{and} \quad \alpha_N = \beta_{N-1} = 0 . \quad (37')$$

Substituting these expressions for x_{N-1} and x_N into the utility function and the first budget constraint in (35) yields the following maximization problem with respect to the remaining variables $x_1, \dots, x_{N-2}$

$$\underset{x_1, \dots, x_{N-2}}{\text{maximize}} \quad u \left[x_1, \dots, x_{N-2}, \frac{\alpha_{kt}}{\alpha_{N-1}} - \sum_{i=1}^{N-2} \frac{\alpha_i}{\alpha_{N-1}} x_i, \frac{\beta_{kt}}{\beta_N} - \sum_{i=1}^{N-2} \frac{\beta_i}{\beta_N} x_i \right] \quad \text{s.t.} \quad \sum_{i=1}^{N-2} \pi_i x_i = y - \Gamma \quad (38)$$

where $\pi_i \equiv p_i - \frac{p_{N-1} \alpha_i}{\alpha_{N-1}} - \frac{p_N \beta_i}{\beta_N}$, and $\Gamma \equiv \frac{p_{N-1} \alpha_{kt}}{\alpha_{N-1}} + \frac{p_N \beta_{kt}}{\beta_N}$.

Solving the maximization in (35) is equivalent to first solving the maximization in (38) and then substituting the resulting demand functions for $x_1, \dots, x_{N-2}$ into (36) to obtain the corresponding demand functions for x_{N-1} and x_N .

Next, I consider a utility maximization similar to (38), but with two extra choice variables and two extra prices

$$\underset{x_1, \dots, x_{N-2}, z_1, z_2}{\text{maximize}} \ u \left[x_1, \dots, x_{N-2}, z_1 - \sum_{i=1}^{N-2} \frac{\alpha_i}{\alpha_{N-1}} x_i, z_2 - \sum_{i=1}^{N-2} \frac{\beta_i}{\beta_N} x_i \right] \quad \text{s.t.} \quad \sum_{i=1}^{N-2} \pi_i x_i + q_1 z_1 + q_2 z_2 = y - \Gamma \quad (39)$$

where π_i and Γ are the same as in (38). The solution to this maximization will be denoted

$x_i = \tilde{h}^i(\cdot), i = 1, \dots, N-2, z_1 = \tilde{h}^{N-1}(\cdot), z_2 = \tilde{h}^N(\cdot)$, and $u = \tilde{v}(\cdot)$. In order to compare the solution to (39) with that of (1), the following extension of Lemma 1 is required:⁴

Lemma 3. Let $x_i = \tilde{h}^i(p, y) \ i = 1, \dots, N$, and $\tilde{v}(p, y)$ be the ordinary demand functions and indirect utility function corresponding to the following utility maximization

$$\underset{x}{\text{maximize}} \ \tilde{u}(x) \equiv u \left(x_1, \dots, x_{N-2}, x_{N-1} - \sum_{j=1}^{N-2} \varphi_j x_j, x_N - \sum_{j=1}^{N-2} \mu_j x_j \right) \quad \text{s.t.} \quad \sum_{i=1}^N p_i x_i = y \quad (40)$$

where $u(\cdot)$ is the same utility function as in (1). These functions are related to the ordinary demand functions and indirect utility functions associated with (1), $x_i = h^i(p, y)$ and $v(p, y)$, as follows

⁴ The proof is given in the Appendix. Note that, if $\varphi_j = 0$ for all $j = 1, \dots, N-2$, then Lemma 3 generates Lemma 1.

$$\tilde{v}(p_1, \dots, p_N, y) = v(\tau_1, \dots, \tau_{N-2}, p_{N-1}, p_N, y) \quad (41a)$$

$$\tilde{h}^i(p_1, \dots, p_N, y) = h^i(\tau_1, \dots, \tau_{N-2}, p_{N-1}, p_N, y) \quad i = 1, \dots, N-2 \quad (41b)$$

$$\tilde{h}^{N-1}(p_1, \dots, p_N, y) = h^{N-1}(\tau_1, \dots, \tau_{N-2}, p_{N-1}, p_N, y) + \sum_1^{N-2} \phi_j h^j(\tau_1, \dots, \tau_{N-2}, p_{N-1}, p_N, y) \quad (41c)$$

$$\tilde{h}^N(p_1, \dots, p_N, y) = h^N(\tau_1, \dots, \tau_{N-2}, p_{N-1}, p_N, y) + \sum_1^{N-2} \mu_j h^j(\tau_1, \dots, \tau_{N-2}, p_{N-1}, p_N, y) \quad (41d)$$

where $\tau_i \equiv p_i + p_{N-1}\phi_i + p_N\mu_i$.

Applying Lemma 3 to (39) yields the following formula for the solution to (39) expressed in terms of the demand functions and indirect utility function associated with (1):⁵

$$\begin{aligned} u &= \tilde{v}(\cdot) = v(\tau_1, \dots, \tau_{N-2}, q_1, q_2, y - \Gamma) \\ x_i &= \tilde{h}^i(\cdot) = h^i(\tau_1, \dots, \tau_{N-2}, q_1, q_2, y - \Gamma) \quad i = 1, \dots, N-2 \\ z_1 &= \tilde{h}^{N-1}(\cdot) = \sum_1^N \frac{\alpha_i}{\alpha_{N-1}} h^i(\tau_1, \dots, \tau_{N-2}, q_1, q_2, y - \Gamma) \\ z_2 &= \tilde{h}^N(\cdot) = \sum_1^N \frac{\beta_i}{\beta_N} h^i(\tau_1, \dots, \tau_{N-2}, q_1, q_2, y - \Gamma) \end{aligned} \quad (42)$$

where $\tau_i \equiv p_i + \frac{\alpha_i}{\alpha_{N-1}}(q_1 - p_{N-1}) + \frac{\beta_i}{\beta_N}(q_2 - p_N)$ and $\Gamma \equiv \frac{p_{N-1}\alpha_{kt}}{\alpha_{N-1}} + \frac{p_N\beta_{kt}}{\beta_N}$.

Next, applying Lemma 2 to (42), one observes that if q_1 and q_2 satisfy

$$\alpha_{kt} = \sum_1^N \alpha_i h^i \left(\tau_1, \dots, \tau_{N-2}, q_1, q_2, y - \Gamma + q_1 \frac{\alpha_{kt}}{\alpha_{N-1}} + q_2 \frac{\beta_{kt}}{\beta_N} \right) \quad (43a)$$

$$\beta_{kt} = \sum_1^N \beta_i h^i \left(\tau_1, \dots, \tau_{N-2}, q_1, q_2, y - \Gamma + q_1 \frac{\alpha_{kt}}{\alpha_{N-1}} + q_2 \frac{\beta_{kt}}{\beta_N} \right) \quad (43b)$$

then the demand functions and indirect utility function associated with (38), expressed in terms of those associated with (1), are given by

⁵ Note that I have taken advantage of the fact that $\alpha_N = \beta_{N-1} = 0$ to simplify the summation so that it runs from 1 to N.

$$x_i = h^j \left(\tau_1, \dots, \tau_{N-2}, q_1, q_2, y - \Gamma + q_1 \frac{\alpha_{kt}}{\alpha_{N-1}} + q_2 \frac{\beta_{kt}}{\beta_N} \right) \quad i = 1, \dots, N \quad (43c)$$

$$u = v \left(\tau_1, \dots, \tau_{N-2}, q_1, q_2, y - \Gamma + q_1 \frac{\alpha_{kt}}{\alpha_{N-1}} + q_2 \frac{\beta_{kt}}{\beta_N} \right). \quad (43d)$$

This may be further simplified by defining

$$\theta_1 \equiv (q_1 - p_{N-1})/\alpha_{N-1} \quad \text{and} \quad \theta_2 \equiv (q_2 - p_N)/\beta_N. \quad (44)$$

The representation theorem for the three constraint problem becomes:

SECOND GLOBAL REPRESENTATION THEOREM.

Given the maximization problem (1) with known solution $x_i = h^i(p, y)$ and $v(p, y)$, the solution to the maximization problem (35) is obtained as follows. Solve

$$\alpha_{kt} = \sum_1^N \alpha_i h^i(p_1 + \theta_1 \alpha_1 + \theta_2 \beta_1, \dots, p_N + \theta_1 \alpha_N + \theta_2 \beta_N, y + \theta_1 \alpha_{kt} + \theta_2 \beta_{kt}) \quad (45a)$$

$$\beta_{kt} = \sum_1^N \beta_i h^i(p_1 + \theta_1 \alpha_1 + \theta_2 \beta_1, \dots, p_N + \theta_1 \alpha_N + \theta_2 \beta_N, y + \theta_1 \alpha_{kt} + \theta_2 \beta_{kt}) \quad (45b)$$

for $\theta_j = \theta_j(p, w, c, y, t, k)$, $j = 1, 2$, where (α_i, β_i) are given by (37). The solution to (35) can then

be expressed as

$$x_i = \bar{h}^i(p, w, c, y, t, k) = h^i(p_1 + \theta_1 \alpha_1 + \theta_2 \beta_1, \dots, p_N + \theta_1 \alpha_N + \theta_2 \beta_N, y + \theta_1 \alpha_{kt} + \theta_2 \beta_{kt}), \quad i = 1, \dots, N \quad (45c)$$

$$u = \bar{v}(p, w, c, y, t, k) = v(p_1 + \theta_1 \alpha_1 + \theta_2 \beta_1, \dots, p_N + \theta_1 \alpha_N + \theta_2 \beta_N, y + \theta_1 \alpha_{kt} + \theta_2 \beta_{kt}). \quad (45d)$$

As in the case of two linear constraints, the properties of the demand functions

$\bar{h}^i(p, w, c, y, t, k)$ can all be derived from (45a-d). For example, given the homogeneity

properties of $v(\cdot)$ and $h^i(\cdot)$, (45a,b) imply that $\theta_1(p, w, c, y, t, k)$ and $\theta_2(p, w, c, y, t, k)$ are each

homogenous of degree 1 in (p, y) , and of degree -1 in (w, t) and also in (c, k) .

Maximization problems with $M \geq 4$ linear constraints can be handled by a straightforward extension of the method shown here for dealing with the three-constraint problem (35). One first takes (M-1) of the constraints, solves them for (M-1) of the x 's as linear functions of the remaining (N-M) x 's, as in (36), and then uses substitution to reduce the original maximization problem to an equivalent problem involving only a single linear constraint and (N-M+1) choice variables, as in (38). One also sets up a parallel maximization problem with a single linear constraint but (M-1) extra choice variables, as in (39). One applies the appropriate analog of Lemma 3 to solve the latter problem, as in (42), and then one applies Lemma 2 to obtain the desired representation of the solution to the original maximization problem in terms of the demand functions and indirect utility function associated with (1), as in (43a-d) or (45a-d). The structure of the representation will be similar to that in (45), with the demand functions taking the form

$$x_i = \bar{h}^i(\cdot) = h^i(p_1 + \theta_1 \alpha_1 + \theta_2 \beta_1 + \theta_3 \gamma_1 + \dots, \dots, p_N + \theta_1 \alpha_N + \theta_2 \beta_N + \theta_3 \gamma_N + \dots, y + \theta_1 \bar{\alpha} + \theta_2 \bar{\beta} + \theta_3 \bar{\gamma} + \dots) \quad (46)$$

where $(\alpha_i, \beta_i, \gamma_i, \dots)$ are functions of the coefficients in the (M-1) linear constraints, $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \dots)$ are functions of both the coefficients and the right-hand-sides of these constraints, and the θ_j are functions of the full set of variables appearing in the M constraints and are defined implicitly through a system of (M-1) equations analogous to (45a,b).

5. Conclusions

The main results of this paper, the Global Representation Theorems for utility maximization problems with two or three linear constraints, show how to derive the ordinary demand functions for these problems using known demand functions that solve the maximization

problem with the same utility function but only a single, linear (budget) constraint. The structure of the demand functions – (19'a) in the case of two linear constraints, and (45c) in the case of three linear constraints – somewhat resembles that of Becker's (1965) model in which the time constraint can be combined with the budget constraint leading to a single constraint involving a “full price” for each commodity consisting of the money price (p_i) plus the monetized value of the time cost, and a “full budget” consisting of exogenous money income plus the monetized value of total available time. The difference is that, here, the several constraints are separately binding, and the parameters which play the role of shadow prices -- θ in the case of (19'a) and (θ_1, θ_2) in the case of (45c) – are endogenous, being themselves functions of the full set of variables that characterize the maximization problem. The demand function structure in (19'a) has already been identified by Larson and Shaikh (2001) as sufficient to satisfy the restrictions on the derivatives of the demand functions which are listed in Proposition 2. The contribution of the present paper is to show that this structure is necessary as well as sufficient.

In terms of computation, while the Global Representation Theorems presented here provide an algorithm for constructing demand functions that correspond to multi-constraint problems involving particular known direct or indirect utility functions, there is no guarantee of obtaining a *closed-form* representation of these demand functions. In fact, it is likely that it will be difficult if not impossible to obtain a closed-form representation of these demand functions. Consider, for example, the Cobb-Douglas utility function

$$u(x) = \prod_1^N x_i^{\gamma_i}, \quad \sum \gamma_i = 1, \quad (47a)$$

for which the standard demand functions take the simple form $x_i = h^i(p, y) = \frac{\gamma_i y}{p_i}$, $i = 1, \dots, N$.

When there is a second linear constraint, as in (7), the ordinary demand functions are now given by

$$x_i = \bar{h}^i(p, w, y, t) = \gamma_i(y + \theta t)/(p_i + \theta w_i) \quad i = 1, \dots, N. \quad (47b)$$

However, as indicated in (18'), θ itself is a function of the parameters of the maximization problem, $\theta = \theta(p, w, y, t)$, since it has to satisfy

$$t = (y + \theta t) \sum_{i=1}^N \gamma_i / (p_i + \theta w_i). \quad (47c)$$

There is no closed form solution to (47c) for θ . Therefore, in this case there is no closed form solution for the demand functions $x_i = \bar{h}^i(p, w, y, t)$.⁶ Nevertheless, one can always use numerical techniques to solve (47c) for θ , and then compute the value of the demand functions in (47b) when (47c) is satisfied. Thus, even when a closed form representation of the demand functions is not available for multi-constraint problems, the Global Representation Theorems should be useful for the numerical calculation of the demands generated by these maximization problems.

⁶ The source of the difficulty is the need for the shadow price function $\theta(p, w, y, t)$ to satisfy (18'). This does not appear to be satisfied by the empirical formula for the shadow value function used recently by Larson and Shaikh (2004), who model θ as a function of a single element of the vector w , say w_N , corresponding to the price of leisure (the wage rate), rather than the full set of variables (p, w, y, t) . Their formula, $\theta = \lambda w_N$, does not appear to satisfy (18') or to possess the homogeneity properties required of $\theta(p, w, y, t)$.

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APPENDIX

Proof of Lemma 1

The maximization in (2) corresponds to the following problem:

$$\begin{aligned} & \underset{x_1 \dots x_{N-1}}{\text{maximize}} \tilde{u}(x_1, \dots, x_N) = u\left(x_1, \dots, x_{N-1}, x_N - \sum_{j=1}^{N-1} \psi_j x_j\right) \\ & \text{s.t.} \quad \sum_{j=1}^N p_j x_j = y \end{aligned} \quad (\text{A})$$

The budget constraint in (A) can be rewritten equivalently as

$$\begin{aligned} y &= \sum_{j=1}^N p_j x_j = \sum_{j=1}^{N-1} (p_j + p_N \psi_j) x_j + p_N \left(x_N - \sum_{j=1}^{N-1} \psi_j x_j \right) \\ &= \sum_{j=1}^{N-1} \pi_j x_j + p_N z \end{aligned}$$

where $z \equiv x_N - \sum_{j=1}^{N-1} \psi_j x_j$ and $\pi_j \equiv p_j + p_N \psi_j$. Next consider the following maximization problem, where $u(\cdot)$ denotes the same utility function as that in (A):

$$\underset{x_1, \dots, x_{N-1}, z}{\text{maximize}} u(x_1, \dots, x_{N-1}, z) \quad \text{s.t.} \quad \sum_{j=1}^{N-1} \pi_j x_j + p_N z = y. \quad (\text{B})$$

Let the solution to this be $(x_{(N)}^*, z^*)$, with the following corresponding ordinary demand functions and indirect utility function:

$$\begin{aligned} x_i^* &= h^i(\pi_1, \dots, \pi_{N-1}, p_N, y) \quad i = 1, \dots, N-1 \\ z^* &= h^N(\pi_1, \dots, \pi_{N-1}, p_N, y) \\ u^* &= v(\pi_1, \dots, \pi_{N-1}, p_N, y). \end{aligned}$$

Problem (B) is essentially the same as the maximization in (1), except that (i) the last argument is here denoted z instead of x_N , and (ii) the prices of the first $N-1$ items are here denoted π_i instead

of p_i . Take the solution to (B) and form $x_N^* \equiv z^* + \sum_1^{N-1} \psi_j x_j^*$. Then, I claim that the vector $x^* \equiv (x_{(N)}^*, x_N^*)$ solves (A); this is equivalent to the statement being made in Lemma 1.

Suppose this claim is not true; I now show this leads to a contradiction.

If the claim is not true, there exists some $x' \neq x^*$ that solves (A),

Note that x^* lies within the budget set of (A), since from (B):

$$\begin{aligned} y &= \sum_1^{N-1} \pi_j x_j^* + p_N z^* \\ &= \sum_1^{N-1} (p_j + p_N \psi_j) x_j^* + p_N z^* \\ &= \sum_1^{N-1} p_j x_j^* + p_N \left(z^* + \sum_1^{N-1} \psi_j x_j^* \right) \\ &= \sum_1^{N-1} p_j x_j^* + p_N x_N^* . \end{aligned}$$

Assume that (A) has a unique maximum. Since x^* lies within the budget set of (A) but does not solve (A), it follows that:

$$u(x^*) < u(x') .$$

But it is also true that x' lies within the budget set of (B), since

$$y = \sum_1^N p_j x_j' = \sum_1^{N-1} \pi_j x_j' + p_N \left(x_N' - \sum_1^{N-1} \psi_j x_j' \right) .$$

Assume that (B) has a unique maximum. Since x' lies within the budget set of (B) but is different from the vector x^* that *does* solve (B), it follows that:

$$u(x') < u(x^*) .$$

Hence there is a contradiction.

Proof of Lemma 2

Suppose that, in (4a), z is fixed at $\bar{z}$, and let x^* denote the corresponding solution to (4a). Let (x', z') denote the solution to:

$$\underset{x,z}{\text{maximize}} \quad u(x,z) \quad \text{s.t.} \quad \sum p x + \sum q z = y + \sum q \bar{z} \quad (C)$$

Observe that if q on the right hand side of (C) takes the value $\bar{q} \equiv \hat{q}(p, \bar{z}, y)$, then $z' = \bar{z}$; let $\bar{x}$ be the corresponding value of x' . Lemma 2 states that $x^* = \bar{x}$.

Suppose this is not true, so that $x^* \neq \bar{x}$. I now show that this leads to a contradiction.

Observe that $(\bar{x}, \bar{z})$ lies within the budget set of (4a) since

$$\sum p \bar{x} + \sum \bar{q} \bar{z} = y + \sum \bar{q} \bar{z} \rightarrow \sum p \bar{x} = y.$$

Assuming that (4a) has a unique solution, this implies that $u(\bar{x}, \bar{z}) < u(x^*, \bar{z})$.

However, $(x^*, \bar{z})$ lies within the budget set of (C), since

$$\sum p x^* = y \rightarrow \sum p x^* + \sum \bar{q} \bar{z} = y + \sum \bar{q} \bar{z}.$$

Assume that (C) has a unique maximum; this implies that $u(x^*, \bar{z}) < u(\bar{x}, \bar{z})$, which yields a contradiction.

Proof of Lemma 3

Observe that the budget constraint in (40) can be written:

$$\begin{aligned} y &= \sum_1^N p_i x_i \\ &= \sum_1^{N-2} (p_i + p_{N-1} \phi_i + p_N \mu_i) x_i + p_{N-1} (x_{N-1} - \sum_1^{N-2} \phi_i x_i) + p_N (x_N - \sum_1^{N-2} \mu_i x_i) \\ &\equiv \sum_1^{N-2} \pi_i x_i + p_{N-1} (x_{N-1} - \sum_1^{N-2} \phi_i x_i) + p_N (x_N - \sum_1^{N-2} \mu_i x_i). \end{aligned}$$

Compare (40) with the following maximization problem

$$\underset{x_1, \dots, x_{N-2}, z_1, z_2}{\text{maximize}} \quad u(x_1 \dots x_{N-2}, z_1, z_2) \quad \text{s.t.} \quad \sum_1^{N-2} \pi_j x_j + p_{N-1} z_1 + p_N z_2 = y. \quad (E)$$

The claim is that, when $z_1 \equiv x_{N-1} - \sum_1^{N-2} \phi_i x_i$ and $z_2 \equiv x_N - \sum_1^{N-2} \mu_i x_i$, the maximization problems in (40) and in (E) are equivalent. If one were to suppose that the claim is not true, this would lead to a contradiction along the lines shown above in the proof of Lemma 1.